

## THE CONSTRUCTION OF TRIGONOMETRIC AND INVERSE TRIGONOMETRIC FUNCTIONS

Rafael Martínez-Planell

University of Puerto Rico at Mayaguez  
rafael.martinez13@upr.edu

Angel Cruz Delgado

University of Puerto Rico at Mayaguez  
angel.cruz14@upr.edu

*This study uses APOS Theory to study students' construction of the sine and cosine functions and their inverses. The conjecture of the constructions students would need to do in order to attain a process conception of these functions is tested by using semi-structured interviews with a group of 11 undergraduate students. Analysis of the interviews suggests a small number of basic mental constructions that students would need to do, that account for observed difficulties, and that hence would need to be stressed during instruction.*

**Keywords:** High School Education, Instructional Activities and Practices, Algebra and Algebraic Thinking

The trigonometric functions and their inverses are important enough in mathematics and its applications to warrant more attention than they have received in the mathematics education research literature. In this paper we use APOS Theory to discuss how students may construct the sine and cosine and their inverse functions. Our research questions are:

1. What basic mental constructions can be conjectured that students need to do in order to have a process conception of the sine and cosine functions and their inverses?
2. Which of the conjectured basic mental constructions students can be observed in students' work when they use these functions in problem solving activities?
3. Which of the conjectured basic mental constructions seem to be lacking when students use these functions in problem solving activities?

### Background

While there is an increasing amount of publications in the mathematics education research literature that deal with trigonometry, due to space restrictions we will not include a review of that literature but rather only mention and refer to three articles.

Bagni (1997) observed in an example that a sizable number of students have trouble finding all solutions of  $\sin x = 1/3$ . Also, for some students the value of the trigonometric functions have no clear sense at real numbers other than the integer multiples of  $\pi/6$ ,  $\pi/4$ , and  $\pi/3$ . He relates these observations to the effect of the didactical contract on students. Weber (2005) argued that an instructional path where trigonometric operations are understood as geometric processes to be both physically performed and reflected upon may result in deeper conceptual understanding of these operations as functions. In her doctoral thesis, Brown (2005) examined the transition from right triangle trigonometry to trigonometry in the coordinate plane, and from there to the graphical representation of sine and cosine. She found that many students had a fragmented or incomplete understanding of sine and cosine, with many not connecting the right triangle, unit circle, and graphical representations. While in her exploration and discussion she seems to favor starting with a ratio definition of trigonometric function and moving from there to the unit circle, we adopt a different approach, as the emphasis of our study falls heavily on the unit circle representation. This will enable us, among other things, to obtain information on students' notion of inverse trigonometric functions, a topic omitted by Brown. While we agree

with many of her observations and recommendations, our approach leads us to consider different issues and constructions in students' learning of trigonometry.

### Theoretical Perspective

Action-Process-Object-Schema (APOS) theory (see for example, Dubinsky and McDonald, 2001), is a well known theory and hence we only give a brief description. An action is a transformation of a mathematical object by individuals, following explicit step by step instructions or using a memorized fact or procedure. An action is perceived as external by the individual. As an action is repeated and the individual reflects on the action it can be *interiorized* into a process. As a process, the transformation may be done mentally; it may be reversed and may be coordinated with other processes. The outcome of the process may be imagined without actually having to perform it. In contrast to an action, a process is perceived as internal. When a person reflects on actions applied to a particular process, the person may become aware of the process as a totality. In this case it is said that the process is *encapsulated* into an object. A mathematical schema is a coherent collection of actions, processes and objects, and other previously constructed schemas, which are synthesized to form mathematical structures utilized in problem situations (Baker, Cooley, & Trigueros, 2000). It is important to note that although it might be thought that in APOS theory there is a linear progression from action to process to object and then to having different actions, processes, and objects organized in schemas, this often appears more like a dialectical progression where there can be partial developments, passages and returns from one to another conception (Czarnocha, Dubinsky, Prabhu, & Vidakovic, 1999). What the theory states is that the way a student works with diverse mathematical tasks related to the concept is different depending on his or her conception.

The application of APOS theory to describe particular constructions by students requires researchers to develop a genetic decomposition – this is a conjecture of specific mental constructions students may make in understanding mathematical concepts and their relationships. Some clarifications are pertinent. The initial genetic decomposition is based on the researchers' knowledge of mathematics, teaching experience, history of the mathematical development of the concept, and any available data. A genetic decomposition for a concept is not unique, it is a model of how a concept may be constructed; different researchers can develop diverse genetic decompositions for a particular concept, but, once one is proposed, it needs to be supported by research data from students. Frequently, data may reveal mental constructions which students make that are different from the ones conjectured, as well as specific conjectured constructions that cause special difficulty for students. This may lead to the development of learning activities for students to help them make the necessary constructions and/or to a revision of the genetic decomposition to better reflect the constructions that students actually make. Eventually the cycle of data gathering, classroom activity implementation, and genetic decomposition revision, stabilizes in a genetic decomposition that serves both to describe the constructions that students actually make as well as to guide instruction.

While we will not include our complete initial genetic decomposition in this paper, we will outline some of the constructions, particularly some specific processes that form part of it.

#### The Process $t \rightarrow P(t)$

One of the processes needed in the construction of sine, cosine and their inverse functions is the process of taking a given real number  $t$  (preferably not an integer multiple of  $\pi/4$  or  $\pi/6$ , at the beginning) and locating the terminal point  $P(t)$  of the unit circle that corresponds to  $t$ . Building this process requires repeating and reflecting upon actions such as those suggested by

Weber (2005), a version of which appears in Figure 1. A student able to imagine doing this without having to explicitly do it can be said to have the process denoted  $t \rightarrow P(t)$ .

### Symmetries of the Circle as a Process

The student will also need to interiorize into a process the action of given a point  $P(t)$ , represented geometrically or as an ordered pair, finding the other three corresponding points  $P(-t)$ ,  $P(t + \pi)$ , and  $P(\pi - t)$  on the unit circle, in the respective representation, geometrically or as ordered pairs. Further actions that students need to interiorize include: given  $t$ , locate any point of the form  $P(t + n2\pi)$ ,  $P(-t + 2n\pi)$ ,  $P(t + \pi + 2n\pi)$ ,  $P(\pi - t + 2n\pi)$  as well as  $P(t + n\pi/2)$ , for integer  $n$ , and find their coordinates, given those of  $P(t)$ . This process plays an important role in the reversal process used to define inverse trigonometric functions. We will refer to it as the “symmetry of the circle” process. Explicit use of these symmetries while solving trigonometric equations may be regarded as evidence that a student has interiorized these actions into a process.

### Process of Projection

Now the sine and cosine functions may be defined by following the  $t \rightarrow P(t)$  process with a process of projection onto the  $y$  axis (to define the sine function), or onto the  $x$  axis (to define the cosine function). We will refer to the concatenation of these two processes as the definitions of the basic trigonometric functions. A process conception requires being able to imagine this concatenation. What we are stressing is that the definitions of the sine and cosine functions require two separate steps: given a real number, assigning to it the coordinates of a point on the unit circle, and then following that process with a projection onto the appropriate axis.

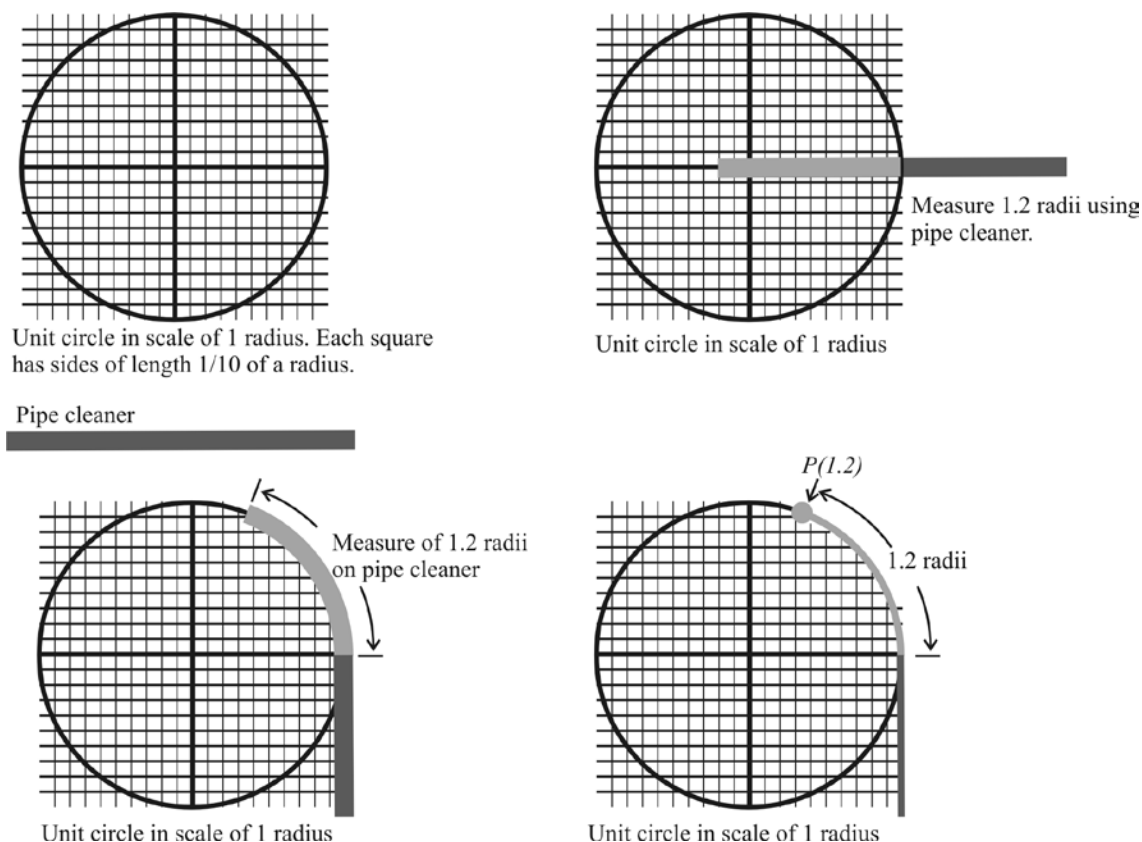
While it is true that formally, anyone defining the trigonometric functions starting with a unit circle approach would at least implicitly use the above decomposition, we are underscoring here that there are three main ideas that would merit being stressed during instruction: the process of given a real number, assigning to it a point on the unit circle; using the symmetries of the circle; and that the definition of the sine and cosine functions follow a two step process (locating a point on the circle followed by a projection). In keeping with Bagni (1997) and Weber (2005), the above constructions need be done for real numbers, not only for integer multiples of  $\pi/6$  and  $\pi/4$ . After having interiorized the process of definition of the sine and cosine functions, the process may be coordinated with the use of technology, allowing students to, among other things, compute values of the sine and cosine functions with the precision afforded by the technological artifact used.

### Reversal of the Process of Projection

Now, the process of definition of the basic trigonometric functions will need to be reversed in order to define the inverse trigonometric functions. Note a process conception of the definition is needed in order to reverse it. Hence one needs a process of reversal of the appropriate projection. The process starts with a number which is either an  $x$  or  $y$  coordinate of a point on the unit circle, produces a geometric representation of the number as a point on the  $x$  or  $y$  axis, and then produces the geometric representation on the unit circle of the corresponding point or points having that  $x$  or  $y$  coordinate. The symbolic representation of those points as an ordered pair of real numbers could be obtained coordinating the points obtained from the projection reversal process with the equation of the unit circle, or, equivalently, with the Theorem of Pythagoras.

After reversing the projection, to obtain one or two points (as geometric entities) on the unit circle, the process  $t \rightarrow P(t)$  needs to be reversed in order to obtain a number  $t$  determining one of the points. This reversal needs to be an interiorized mental process, and may be constructed

with activities analogous to the ones in Weber (2005), or equivalently, using the unit circle and pipe cleaner as suggested by Figure 1. This process reversal may be done with geometric arguments in the case of special values. Observe that memorizing tables of special values will only result in an action conception, not in the needed mental process. The action of obtaining one real number  $t$  corresponding to a given point  $P(t)$  of the unit circle, will now need to be coordinated with the symmetry process to produce all real numbers  $t$  determining the corresponding points.



**Figure 1: Using a Pipe Cleaner to Locate Point  $P(1.2)$  on the Unit Circle.**

The chain of actions that starts with a number (represented as an  $x$  or  $y$  coordinate), goes on to identify the point or points on the unit circle having that number as an  $x$ , or respectively  $y$  coordinate, and then identifies all the real numbers corresponding to the point or points on the unit circle, may be interiorized into a process that we call reversal of the definition. This process starts with a coordinate and produces the collection of all real numbers corresponding to the points on the unit circle (one, two, or none) having that coordinate. By its nature, this process does not define a function.

### The Process of Range

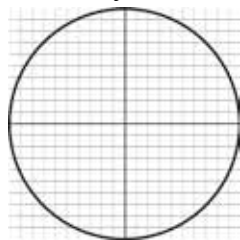
The action of coordinating the above definition reversal process with the function schema is needed in order to have the inverse trigonometric functions. In particular, the coordination of the process of reversing the definition with, what we will call, the process of Range (of the corresponding inverse trigonometric function), is interiorized into a process of inverse trigonometric function. As will be seen, an action conception of Range (having memorized the

range of the inverse sine and cosine) will not enable students to build a process conception of inverse trigonometric functions. We conjecture that building a process conception of Range requires explicit attention and will not follow from the mere statement of a formal definition. Students will need to interiorize actions that explore ways of restricting the domain of the sine and cosine functions so that the resulting function is one to one and the restricted domain is connected and as large as possible. These actions should include both, unit circle representation and the graphs of these functions. The process resulting from the interiorization of these actions would allow students to argue for the need and also the convenience of restricting the domains of sine and cosine as they normally are. Otherwise, students will be constrained to having an action conception of Range as a memorized fact. The coordination of the process of exploration of possible restrictions of the domains of sine and cosine with the concept of range of an inverse function is what we call the Range process (of an inverse trigonometric function). Once students have a process of inverse trigonometric functions (coordination of definition reversal and Range), it may be reinforced coordinating it with the use of technology through actions that explore and compare the mental constructions that have been made.

### Method

The study took place at a mid-size public university in Puerto Rico (approximately 14,000 students) with a large science and engineering component (at least 60% of the students). Semi-structured interviews were conducted with 11 students in order to test a preliminary genetic decomposition. Of the students interviewed, 3 were above average, 5 average, and 3 below average according to their professor. Students were chosen this way in order to be able to observe a wide range of different behaviors when engaged in mathematical tasks. All students were enrolled in the same section of a precalculus course, where they used a traditional textbook (Stewart, Redlin, and Watson, 2011) and received traditional instruction. The interviews were recorded and transcribed, and records of students' written responses were kept. The interviews were analyzed independently by the researchers, by comparing student constructions to those predicted by a preliminary genetic decomposition. They were then discussed, and differences were negotiated. Student performance on the interview questionnaire was graded and scores were kept. Seven of the 12 interview questions were pertinent to the present article:

1. Use the unit circle in the figure and the pipe cleaner [a string, as in Figure 1] to approximate the value of  $\sin(1.2)$  the best you can.

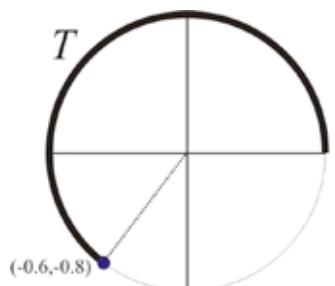


2. Use the unit circle in the figure and the pipe cleaner to approximate the value of  $\sin^{-1}(0.8)$  as best you can.
3. Given that  $\sin\left(\frac{10\pi}{7}\right) = -0.974927912$ . Find  $\sin^{-1}(-0.974927912)$ .
4. Draw on a unit circle *all* terminal points  $P(t)$  corresponding to numbers  $t$  of the form

$$4t = \frac{\pi}{3} + 2n\pi, n \text{ an integer, and that are between } 0 \text{ and } 2\pi.$$



5. Find all solutions of  $\cos(t) = -\frac{1}{2}$  in  $[0, 2\pi)$ .
6. Find all solutions of  $\sin(t) = -\frac{3}{5}$ .
7. We start on point  $(1, 0)$  on the unit circle and travel counterclockwise on the circle a distance  $T$ , as shown in the figure below, ending at point  $(-0.6, -0.8)$ . What is the value of  $T$ ?



### Results

#### The Process $t \rightarrow P(t)$

Recall this is the process of assigning a point on the unit circle to any given real number. Of the interviewed students, 8 of 11 had difficulty assigning a point to a number.

When originally asked to estimate the value of  $\sin(1.2)$  using the pipe cleaner and the unit circle, and after having shown Ricky how to use the pipe cleaner to measure arc-length on the unit circle (as in Figure 1):

*Interviewer:* Is there some way you can use the pipe cleaner to estimate  $\sin(1.2)$ ?

*Ricky:* I don't think so, I don't know... It is finding 1 and 1.2 [he is referring to finding  $P(1)$  and  $P(1.2)$ ] and I'm finding it difficult ... well, sine is y, ... drawing a line towards y [referring to the y axis]

Later on, after a long intervention where the interviewer helped Ricky locate  $P(1.2)$ :

*Ricky:* So, sine of 1.2, what comes to mind is ... when one says that sine is y ... is drawing a line to the y axis and that would give me a value.

Note that Ricky has difficulty with the  $t \rightarrow P(t)$  process (seems to have an action conception) but seems to be aware of the need to project.

#### Symmetries of the circle as a process

Six of the 11 interviewed students had difficulty using the symmetries of the circle. Maria, the student with the second best performance in the interview questionnaire (71%) is an example of a student not having interiorized the use of circle symmetries as a process. In Problem 7, after using a calculator to compute  $\cos^{-1}(-0.6)$  she needs to find the corresponding point on the third quadrant:

*Maria:* Use this angle ... to do something, the answer then is... add pi.

#### The Process of Projection

We saw the case of Ricky, who had not interiorized the  $t \rightarrow P(t)$  process but seemed to recognize the need to follow this with a projection onto the y axis in the definition of the sine function. In the case of Diana, it is the other way around. Diana was the only student not showing any difficulty with either the  $t \rightarrow P(t)$  process or the process of symmetries of the circle, and by far the student having the best all around performance on the interview

questionnaire (92%). Three instances where she failed to immediately obtain a correct response had to do with her failure to do the corresponding projection:

*Diana:* [In problem 1, after correctly using the pipe cleaner to locate  $P(1.2)$  without any help] In this case, sine of 1.2 is equivalent to, the  $y$  value of a terminal point.

*Interviewer:* How would you estimate  $\sin(1.2)$ ? You may use the pipe cleaner as you see fit.

*Diana:* I don't know if to apply  $a^2 + b^2 = c^2$  ... I don't know, nothing comes to mind ...

Latter when doing Problem 7, Diana again fails to project onto an appropriate axis (in order to use either  $\cos^{-1}(-0.6)$  or  $\sin^{-1}(-0.8)$ ):

*Diana:* ... What is the value of  $T$ ? ... one would need to find ... the measure of the angle between  $\pi$  and the terminal point and add to it ... ay! It seems to be so easy!

*Interviewer:* What are you thinking?

*Diana:* I'm thinking in ... it doesn't occur to me .... [She was eventually able to use right triangle trigonometry and the inverse tangent function to successfully complete the problem]

### Reversal of the Process of Definition and Inverse Trigonometric Functions

All 11 students interviewed failed at one point or another, to apply an inverse trigonometric function. Indeed, of the 12 questions in the interview questionnaire, the four problems in which students achieved the lowest overall score were problems directly dealing with the inverse sine or inverse cosine functions (36% in Problem 2, 13.6% in Problem 3, 31.8% in Problem 6, and 30% in Problem 7). In the interviews we observed students who showed difficulty using an inverse trigonometric function due to different reasons: not having interiorized the  $t \rightarrow P(t)$  process, or the projection process, or the circle symmetries, or the Range process.

For example, Gina seems to not have interiorized the projection into a process (in problem 2), hence she shows difficulties with its reversion:

*Gina:* Well I have that sine represents the  $y$ ... [She goes on to measure 0.8 correctly on the pipe cleaner but locates  $P(0.8)$  rather than a point with  $y$  coordinate equal to 0.8].

Gina is then directed to compute  $\sin^{-1}(0.8)$  on the calculator but she still fails to relate this to the 0.8 on the  $y$  axis:

*Gina:* ... the angle of 53 degrees would give the point  $P(t)$ .

*Interviewer:* I'd like to see that angle, where is it? [Referring to the unit circle in the figure used for problems 1 and 2.]

*Gina:* Well, it would be around here, I would draw an approximate radius, since I have nothing to measure it with [Even though a horizontal line crossing the  $y$  axis at 0.8 is part of the figure, she does not use it or refer to it].

Another common observation is that students may be able to do an action or process of reversing the definition when working in the context of special values, like Problem 5, and yet fail to use the process of reversing a definition in the more general context, as in Problem 6. Indeed, 6 students had problem 5, either partially or completely, correct and problem 6 completely incorrect. This suggests that the actions being interiorized by students should be carefully chosen to include the general context as well as the context of special values.

It is important and interesting to notice that none of the students used a drawing in trying to make sense of problems 5 and 6 (solving  $\cos(t) = -1/2$  and  $\sin(t) = -3/5$ ). This stresses the need and importance of the actions that give explicit attention to the geometric representation of the unit circle when reversing the process of definition of sine and cosine.

Observed results sustain the need to help students develop the conjectured constructions.

### Summary

In order to construct a sine or cosine function, students need a process ( $t \rightarrow P(t)$ ), where they start with a real number and assign to it a point on the unit circle, and they need to follow that process with a projection onto the appropriate axis. Hence, we argue that instruction would need to emphasize that the definitions of these functions follow a two-step process and it would also need to emphasize that the definition applies to any real number, not only special values. It was found that the two steps in this construction, the process ( $t \rightarrow P(t)$ ) and the projection, are two independent processes that account for many student mistakes in problem solving situations.

We also observe that the construction of an inverse trigonometric function requires essentially, reversing a definition. Reversal of the definition process, consists of reversing the projection by starting with an  $x$  or  $y$  coordinate of a point on the unit circle and ending with one or two points on the unit circle, followed by reversing the process  $t \rightarrow P(t)$  and coordinating with the symmetries of the unit circle to obtain all real numbers determining a point on the unit circle having the given  $x$  or  $y$  coordinate. In particular, this suggests that instruction would need to emphasize activities that lead students to interiorize actions exploring the symmetries of the circle. It was also conjectured that an inverse trigonometric function results from the coordination of the definition reversal process with the appropriate Range process. It was found that student' difficulties, particularly in problems dealing with inverse trigonometric functions resulted from lacking some of the aforementioned processes. It could also be deduced that instruction would need to aim at having students build a process of Range for the inverse trigonometric functions, relating unit circle and graphical arguments of why the sine and cosine functions are restricted they way they are when defining inverse trigonometric functions. It was observed that not having this process, students are left with a memorized fact, hence an action conception of Range, making it difficult to recall and use the range of an inverse trigonometric function in problem situations. It was clear that the notion of an inverse trigonometric function is a complex construction that accounts for many student mistakes in problem solving situations.

Much work still needs to be done: developing activities to help students do the conjectured constructions, class testing, performing a second round of interviews including students' use of graphical and numerical methods in problem solving, and refining the genetic decomposition accordingly. Nevertheless, the conjectured process constructions seem useful in describing student behavior in problem solving, in organizing research, and in helping guide instruction.

### References

- Bagni, G.T. (2007). Trigonometric functions: Learning and didactical contract. In B. D'Amore & A. Gagatsis (Eds.), *Didactics of Mathematics-Technology in Education* (pp. 3-10). Retrieved from <http://www.syllogismos.it/giorgiobagni/publications.htm>
- Baker, B., Cooley, L., & Trigueros, M. (2000). The schema triad - a calculus example. *Journal for Research in Mathematics Education*, 31, 557-578.
- Brown, S. (2005) *The trigonometric connection: students' understanding of sine and cosine*. (PhD.thesis), Illinois State University. Retrieved from ProQuest, UMI Dissertations Publishing, thesis number 3233908.
- Czarnocha, B., Dubinsky, E., Prabhu, V., & Vidakovic, D. (1999). One theoretical perspective in undergraduate mathematics education research. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of PME* (Vol. 1, pp. 95-110). Haifa, Israel: PME.
- Dubinsky, E. & McDonald, M. A. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In, D. Holton at al (Eds.), *The teaching and learning of mathematics at university level: An ICMI study* (pp. 273-280). Dordrecht, The Netherlands: Kluwer Academic Press.
- Stewart, J., Redlin, L., & Watson, S. (2011). *Precalculus*. United States: Thompson.
- Weber, K. (2005). Students' understanding of trigonometric functions. *Mathematics Education Research Journal*, (17)3, 91-112.